Journal of Statistical Physics, Vol. 72, Nos. 5/6, 1993

Some Remarks on Discrete Aperiodic Schrödinger Operators

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Received January 11, 1993

We consider Schrödinger operators on $l^2(\mathbb{Z}^\nu)$ with deterministic aperiodic potential and Schrödinger operators on the l^2-space of the set of vertices of Penrose tilings and other aperiodic self-similar tilings. The operators on $l^2(\mathbb{Z}^\nu)$ fit into the formalism of ergodic random Schrödinger operators. Hence, their Lyapunov exponent, integrated density of states, and spectrum are almost-surely constant. We show that they are actually constant: the Lyapunov exponent for one-dimensional Schrödinger operators with potential defined by a primitive substitution, the integrated density of states, and the spectrum in arbitrary dimension if the system is strictly ergodic. We give examples of strictly ergodic Schrödinger operators that include several kinds of "almost-periodic" operators that have been studied in the literature. For Schrödinger operators on Penrose tilings we prove that the integrated density of states exists and is independent of boundary conditions and the particular Penrose tiling under consideration.

KEY WORDS: Discrete Schrödinger operators; Lyapunov exponent; integrated density of states; spectrum; Fibonacci sequences; primitive substitutions; Penrose tilings; self-similar tilings; strict ergodicity; unique ergodicity; minimality; aperiodic structures.

1. INTRODUCTION

The aperiodic Schrödinger operators we deal with are Schrödinger operators on $l^2(\mathbb{Z}^\nu)$ with a deterministic aperiodic potential and Schrödinger operators on the l^2-space of the set of vertices of Penrose tilings or other self-similar aperiodic tilings. In both cases the Schrödinger operator H is of the form

$$(H\psi)(x) = \sum_{\langle x, y \rangle} \psi(y) + V(x)\,\psi(x) \tag{1.1}$$

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0022-4715/93/0900-1353$07.00/0

where the summation extends over the nearest neighbors of x. These operators have received a lot of attention recently (for references see Sections 6 and 7.3). They are studied because they might describe electronic properties[1-3] of quasicrystals,[4] because they are relevant to quasiperiodic phenomena in solid-state physics (see, e.g., ref. 5), and because they are mathematically interesting.

The aperiodic Schrödinger operators on $l^2(\mathbb{Z}^\nu)$ fit into the formalism of ergodic random Schrödinger operators (see, e.g., refs. 6–8) in the following canonical way. Let Ω denote the closure in a suitable topology of the set of all translates of the potential V. The space Ω is a compact metrizable topological space that is given its Borel σ-algebra $\mathscr{F}$. Every $\omega \in \Omega$ defines a potential $V(\omega)$ by $V_n(\omega) := \omega_n$. The potential is a random variable on the measurable space $(\Omega, \mathscr{F})$ and one gets a random Schrödinger operator H_ω. Let $\mathbb{Z}^\nu$ act on Ω by shifts T_i defined by $(T_i\omega)_n := \omega_{n+i}$ and on $l^2(\mathbb{Z}^\nu)$ by unitary operators U_i defined by $(U_i\psi)_n := \psi_{n+i}$. The family H_ω satisfies

$$H_{T_i\omega} = U_i^* H_\omega U_i \tag{1.2}$$

There is a natural probability measure μ on $(\Omega, \mathscr{F})$ that is ergodic with respect to the shifts T_i.

For ergodic random Schrödinger operators, the spectrum, the integrated density of states, and (in one dimension) the Lyapunov exponent exist μ-almost surely and are μ-almost surely constant (see, e.g., refs. 6–8). This paper will show that these quantities actually exist for all ω and do not depend on it, under the following conditions.

If the system is strictly ergodic (i.e., uniquely ergodic and minimal; see Section 7.1 for definitions), then the integrated density of states and the spectrum of H_ω are independent of ω (Propositions 7.3 and 7.4, respectively). Examples of strictly ergodic Schrödinger operators are given in Section 7.3. They include several kinds of "almost-periodic" Schrödinger operators that have appeared in the literature, and some new ones, too. The Lyapunov exponent is not independent of ω for arbitrary strictly ergodic Schrödinger operators. However, Proposition 5.1 states that the Lyapunov exponent is independent of ω if the potential derives from a so-called primitive substitution. Examples of sequence derived from primitive substitutions are Thue–Morse sequences and Fibonacci sequences.

Schrödinger operators on the l^2-space of a Penrose tiling do not fit into the formalism of ergodic random Schrödinger operators because Penrose tilings are aperiodic. The absence of periodicity is a serious obstacle for analytical or rigorous results. As far as we know, Proposition 6.1 is the first rigorous result on these models. It states that the integrated density of states exists for a large class of Schrödinger operators on Penrose tilings and is independent of boundary conditions. Moreover, if one considers the same (in a sense to be specified later) Schrödinger operator on a different Penrose tiling, then the integrated density of states does not change. (Recall that there are uncountably many different Penrose tilings.) The result is of interest because it shows that in numerical calculations of the integrated density of states the "choice of the initial seed for the lattice,"[9] which amounts to a choice of the Penrose tiling, does not affect the results in the thermodynamic limit. The proof of Proposition 6.1 uses the self-similarity of the Penrose tilings. It works for Schrödinger operators on every self-similar tiling in every dimension.

The structure of the paper is as follows. Section 2 provides the necessary background on primitive substitutions. Section 3 discusses Penrose tilings as examples of self-similar aperiodic tilings. Section 4 states a theorem on the existence of the spatial mean of subadditive set functions. It is used in Section 5 to prove that the Lyapunov exponent is independent of the realization of the potential if the potential derives from a primitive substitution. In Section 6 the same theorem is applied to prove the existence of the integrated density of states for Schrödinger operators on Penrose tilings. Section 7 considers strictly ergodic Schrödinger operators on $l^2(\mathbb{Z}^\nu)$; Section 7.1 contains the definitions, Section 7.2 the results. Section 7.3 gives three classes of examples of strictly Schrödinger operators: those with potential defined by a primitive substitution, by a "circle map," or by a uniformly almost-periodic function.

2. PRIMITIVE SUBSTITUTIONS

Primitive substitutions define two-sided infinite sequences taking finitely many values. Often these sequences are aperiodic, but even then they have very good homogeneity properties. The orbit closure under the shift of such a sequence gives a so-called substitution dynamical system. There is an extensive literature on substitution dynamical systems (see ref. 10 and references contained therein). This section explains how primitive substitutions define two-sided infinite sequences and states some properties of these sequences and of substitution dynamical systems.

Let A be a finite set. It is called an alphabet and its elements are called symbols. A finite sequence of symbols is called a word. The set of all words is denoted by A^*. An example would be $A = \{0, 1\}$; then '0', '01', and '000' would be examples of words. A substitution (on A) is a map $S: A \to A^*$; it will be extended to a map $A^* \to A^*$ and $A^{\mathbb{N}} \to A^{\mathbb{N}}$ by concatenation, i.e., $S(a_1 a_2 \cdots a_n) := (Sa_1)(Sa_2) \cdots (Sa_n)$. A substitution is called primitive if there is an integer k such that for all symbols a the word $S^k a$ contains at least one copy of every symbol. An example of a primitive substitution on $\{0, 1\}$ is

$$\begin{aligned} S_F 0 &:= 01 \\ S_F 1 &:= 0 \end{aligned} \tag{2.1}$$

This substitution defines the so-called Fibonacci sequences. Clearly, $S_F^k 0 = S_F^{k-1} 0 S_F^{k-1} 1$ for all k. Since $S_F^k 0$ starts with $S_F^{k-1} 0$, the substitution S_F has a fixed point in $\{0, 1\}^{\mathbb{N}}$. It will be denoted by ω_F^+. Often, ω_F^+ is called *the* Fibonacci sequence. The fixed point arises from the fact that $S_F 0$ starts with 0. Similarly, if there is a symbol $a \in A$ such that Sa starts with the symbol a, then a defines a fixed point of S on $A^{\mathbb{N}}$, which will be denoted by ω^+. The existence of such a symbol can be assumed without loss of generality because the alphabet is a finite set. Hence for every $b \in A$ there will be at least two integers $k \geqslant 0$ and $j > 0$ such that $S^k b$ and $S^{k+j} b$ both start with the same symbol, say a. Then $S^j a$ starts with a and one can consider the substitution $S' = S^j$, which is primitive because S is primitive.

We have explained how a primitive substitution gives rise to a one-sided infinite sequence ω^+. We shall now use ω^+ to define a set $\Omega \subset A^{\mathbb{Z}}$ of two-sided infinite sequences. It is, however, possible to generate two-sided infinite sequences directly by means of the substitution (see, e.g., refs. 11 and 12).

Let $\bar{\omega}$ be any element of $A^{\mathbb{Z}}$ that coincides with ω^+ on the positive integers. Define Ω as the elements of $A^{\mathbb{Z}}$ that are limit points (in the product topology) of $T_n \bar{\omega}$ as $n \to \infty$, i.e.,

$$\Omega := \{\omega \in A^{\mathbb{Z}} \mid \omega = \lim_{i \to \infty} T_{n_i} \bar{\omega} \text{ and } n_i \to \infty\}$$

By definition, Ω is a closed (and therefore compact) subset of $A^{\mathbb{Z}}$ that is invariant under $T := T_1$. The dynamical system (Ω, T) is called the substitution dynamical system associated to S. The set Ω is finite if and only if ω^+ is periodic. If Ω is not finite, then it is uncountably infinite.

The substitution dynamical system (Ω, T) and the sequences in Ω have the following well-known properties (see, e.g., ref. 10). First, the system is uniquely ergodic. This means that there is exactly one ergodic invariant probability measure on Ω. Second, it is minimal, which means that every $\omega \in \Omega$ has the property that its orbit $\{T_k \omega\}_{k \in \mathbb{Z}}$ is dense in Ω. Third, every word that occurs in some $\omega \in \Omega$ occurs infinitely often, and with bounded gaps, in every $\eta \in \Omega$. It even occurs with a well-defined frequency, in the following sense. If $N_w(k, L)$ is the number of times the word w occurs in η in the interval $\{k, k+1, ..., k+L-1\}$, then

$$n_w := \lim_{L \to \infty} \frac{1}{L} N_w(k, L) \tag{2.2}$$

	0	1	0	0	1.	0	1	0	0	1	0	0	1	0	1	0	0	1	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0	1
$\mathcal{T}^1$	0		1	0		0		1	0		1	0		0		1	0		0		1	0		1	0		0		1	0		1	0	
$\mathcal{T}^2$	0			1		0			0			1		0			1		0			0			1		0			0			1	
$\mathcal{T}^3$	0					1			0					0					1			0					1			0				
$\mathcal{T}^4$	0								1					0								0								1				
$\mathcal{T}^5$	0													1								0												
$\mathcal{T}^6$	0																					1												

Fig. 1. Compositions of part of a Fibonacci sequence.

exists uniformly in k and is independent of η. Thus, sequences in Ω are very homogeneous. The fact that for every word w the limit in (2.2) exists uniformly with respect to k for all $\eta \in \Omega$ is equivalent to the unique ergodicity.

Finally, a property that will be very important is the self-similarity of the sequences (cf. the next section). For every $\omega \in \Omega$ there is a uniquely determined sequence $\{\mathcal{T}^j\}_{j=0}^{\infty}$ of so-called compositions. The jth composition $\mathcal{T}^j$ consists of pairs $\{(I_i^j, a_i^j)\}_{i \in \mathbb{Z}}$, where I_i^j is an interval $\{l_i, l_i+1,..., l_{i+1}-1\}$ and a_i^j is a symbol. The dependence of l_i on j is left implicit. The intervals I_i^j form a partition of $\mathbb{Z}$. The sequence $\{a_i^j\}_{i \in \mathbb{Z}}$ is itself an element of Ω that has the following relation to the original sequence ω:

$$S^j a_i^j = \omega_{l_i} \omega_{l_i+1} \cdots \omega_{l_{i+1}-1}$$

In words: if one applies the substitution j times to a_i^j, one gets the word of ω that lies in I_i^j. Note that $\mathcal{T}^0$ can be identified with the sequence ω itself. The construction is illustrated in Fig. 1 for a part of a Fibonacci sequence. Elements (I_i^j, a_i^j) and (I_k^j, a_k^j) of $\mathcal{T}^j$ will be called equivalent if $a_i^j = a_k^j$. In that case the sets I_i^j and I_k^j define two occurrences of the word $S^j a_i^j$.

3. PENROSE TILINGS AS EXAMPLES OF SELF-SIMILAR TILINGS

This section gives the information on Penrose tilings that will be needed to discuss Schrödinger operators on Penrose tilings and to state the theorem in the next section. In particular, it explains the self-similarity of Penrose tilings. There exist many tilings with a self-similarity analogous to that of Penrose tilings.[13] More detailed information on Penrose tilings can be found in refs. 14–17. Penrose tilings have become a standard two-dimensional model of quasicrystalline order.

Figure 2 shows a part of a Penrose tiling. Every tile is a rhomb, and there are two kinds of rhombs: fat ones and skinny ones. There are several

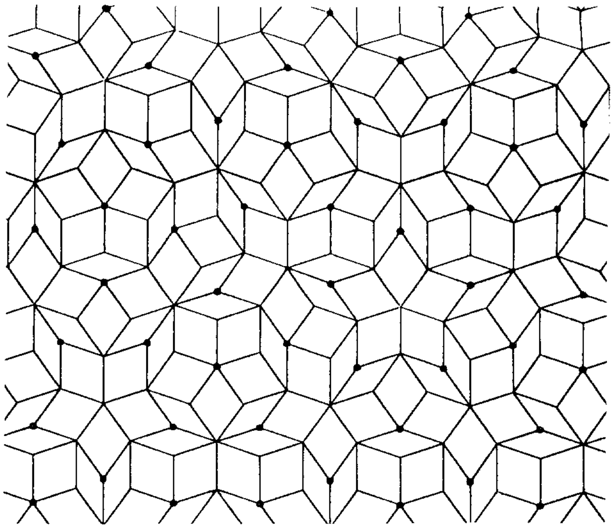

Fig. 2. Part of a Penrose tiling. The dots come from the "matching rules"; see, e.g., ref. 14.

ways to define Penrose tilings, which can be found in the literature.[14–17] Every Penrose tiling is aperiodic: no Penrose tiling will coincide with itself after any nonzero translation. There are uncountably many different Penrose tilings (two Penrose tilings are different if they cannot be made to coincide by a translation and/or rotation). Moreover, Penrose tilings are indistinguishable in the sense that every configuration of tiles (every "patch") that occurs in one Penrose tiling occurs in every other Penrose tiling. Moreover, every patch that occurs in a Penrose tiling occurs with a well-defined, strictly positive frequency that is the same for all Penrose tilings. The limit of the number of copies/volume exists uniformly with respect to the position of the volume.

Penrose tilings have a self-similarity property that is analogous to the self-similarity that was discussed in the previous section. On every Penrose tiling one can superimpose another, uniquely defined, Penrose tiling in which the edges are a factor $\tau = \frac{1}{2}(1+\sqrt{5})$ longer (see Fig. 3). This second tiling is called the composition of the original one. Note that every pair of fat (skinny) rhombs contains the same pattern from the original tiling. Since the composition is a Penrose tiling, there exists a (unique) composition of the composition, which is called the second composition, and so on. Thus, there is a sequence of compositions $\{\mathscr{T}^j\}_{j=1}^{\infty}$ associated with every Penrose tiling $\mathscr{T}$.

There are quite a few examples of aperiodic tilings of the plane and of space that are self-similar in the way Penrose tilings are self-similar (for

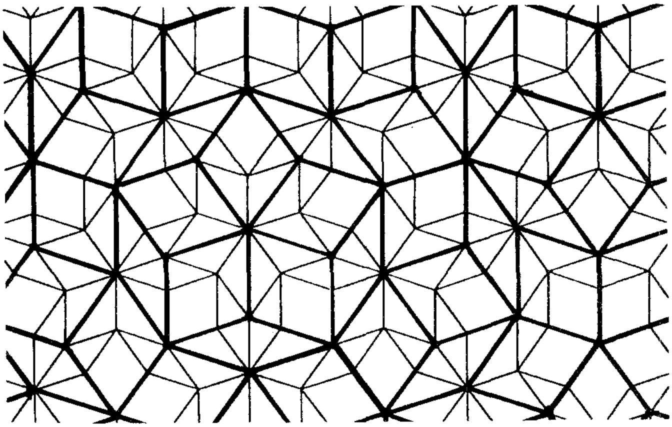

Fig. 3. Part of a Penrose tiling (thin lines) and its first composition (thick lines).

some references see, e.g., p. 171 of ref. 18). We call such tilings self-similar tilings. Lunnon and Pleasants[13] have proved that there exist many different kinds of aperiodic self-similar tilings in every dimension. Every self-similar tiling has its sequence of compositions.

Note that one can view $\mathbb{Z}^\nu$ as the set of vertices of a self-similar tiling: a tiling of space by cubes of side one. The first composition is formed by taking a^ν cubes together to form cubes of side a. Tiles in the jth composition are cubes of side a^j. Here a can be any positive integer. But even for a given integer a compositions are not uniquely determined; one can shift them. The partitions of $\mathbb{Z}$ associated with primitive substitutions in the previous section give a way of viewing $\mathbb{Z}$ as the set of vertices of a self-similar aperiodic tiling, one in which tiles are intervals of length 1 marked by a symbol. One can also view $\mathbb{Z}^\nu$ as the set of vertices of a self-similar aperiodic tiling by marked cubes. As an example,[18] consider the two-dimensional substitution

$$0 \to \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \qquad 1 \to \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

Iterating the substitution on 0 gives a pattern of 0's and 1's in the first quadrant of $\mathbb{Z}^2$. Its limit points under translations yield aperiodic elements of $\{0, 1\}^{\mathbb{Z}^2}$. These can be associated with tilings by marked squares in an obvious way and these tilings are self-similar.

A point on the boundary of a tile belongs to at least one other tile. To formulate the theorem in the next section, however, we need to have a sequence of partitions associated with the compositions. To obtain a partition from $\mathcal{T}^j$ one only has to decide to which set of the partition every boundary point of a tile will belong. This can be done in such a way that two sets from the partition contain the same pattern of tiles of $\mathcal{T}$ whenever these sets are translates of each other (see p. 173 of ref. 18 for a detailed explanation). From now on we shall simply assume that the compositions $\mathcal{T}^j$ are these partitions.

4. THE MEAN OF SUBADDITIVE SET FUNCTIONS

Let $\Lambda \to F_\Lambda$ be a function on the bounded Lebesgue-measurable subsets of $\mathbb{R}^\nu$ that is negative

$$F_\Lambda \leqslant 0 \qquad \text{for all } \Lambda \tag{4.1}$$

and subadditive

$$F_{\Lambda' \cup \Lambda} \leqslant F_{\Lambda'} + F_\Lambda \qquad \text{if} \quad \Lambda' \cap \Lambda = \varnothing \tag{4.2}$$

Let $|\Lambda|$ denote the Lebesgue measure of Λ. The existence of the mean $\lim_{|\Lambda| \to \infty} |\Lambda|^{-1} F_\Lambda$ is well known[19] for the case that F_Λ is translation invariant (or periodic). But in our applications F_Λ will not be translation invariant, because it will for instance be proportional to the integrated density of states of a Schrödinger operator on a Penrose tiling. Theorem 4.1 below gives necessary and sufficient conditions for the existence of the mean. The conditions involve the behavior of F_Λ on the partitions $\mathcal{T}^j$. Before stating the theorem, we first specify the sense in which $|\Lambda| \to \infty$ and define the notion of vertexneighborhood.

A sequence $\{\Lambda_n\}_{n=0}^{\infty}$ of Lebesgue-measurable subsets of $\mathbb{R}^\nu$ tends to infinity in the sense of Van Hove if for $n \to \infty$ both $|\Lambda_n| \to \infty$ and $|\partial_r \Lambda|/|\Lambda_n| \to 0$ for all $r > 0$, where $\partial_r \Lambda$ denotes the set of points of $\mathbb{R}^\nu$ that have a distance of at most r to the boundary of Λ_n. Note that it is *not* required that $\Lambda_n \subset \Lambda_{n+1}$, nor that the union of the Λ_n equals $\mathbb{R}^\nu$. A sequence $\{\Lambda_n\}$ is called a cubelike sequence if (i) it is a Van Hove sequence and (ii) there is a $\delta > 0$ and a sequence of cubes $\{K_n\}$ such that for all n both $\Lambda_n \subset K_n$ and $|\Lambda_n|/|K_n| \geqslant \delta$. Every "Fisher sequence" is a cubelike sequence (Lemma 1 in ref. 20).

A vertexneighborhood of $\mathcal{T}^j$ is the union of all sets of $\mathcal{T}^j$ that meet a vertex, in the sense that their intersection with every sphere around the vertex is nonempty. In the context of Section 2, a vertexneighborhood of $\mathcal{T}^j$ consists of two adjacent sets, i.e., an interval $\{l_i,..., l_{i+2} - 1\}$ and the symbols a_i^j and a_{i+1}^j. Two vertexneighborhoods are equivalent if they can be mapped onto each other by a translation (in the context of Section 2, the symbols should also be the same).

The following theorem has been proved in ref. 18.

Theorem 4.1. Suppose F_Λ satisfies (4.1) and (4.2). If

$$\lim_{j \to \infty} \sup_{\substack{V, V' \in \mathcal{T}^j: \\ V' = V + x}} \frac{1}{|V|} |F_V - F_{V'}| = 0 \tag{4.3}$$

where the supremum is taken over all pairs of equivalent sets and vertexneighborhoods in $\mathcal{T}^j$, then there exists an $f \in [-\infty, 0]$ such that

$$f = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} F_{\Lambda_n} \qquad \text{for every cubelike sequence } \{\Lambda_n\} \tag{4.4}$$

Conversely, if f is finite, then (4.4) implies (4.3).

Condition (4.3) is also equivalent to

$$\lim_{a \to \infty} \sup_{x \in \mathbb{R}^\nu} a^{-\nu} |F_{C_a} - F_{C_a + x}| \tag{4.5}$$

where C_a denotes a cube of side a. So F_Λ may fluctuate when Λ is shifted around, as long as the fluctuations grow slower than $|\Lambda|$ as $|\Lambda|$ becomes large. The convergence of $|\Lambda|^{-1} F_\Lambda$ will then be uniform with respect to the position of Λ.

Instead of dividing in (4.4) by the Lebesgue measure of Λ_n, one may also divide by the number of lattice points in Λ_n, or, in the case of an aperiodic self-similar tiling, by the number of vertices in Λ_n. In fact, this is what shall be done in the rest of the paper: Λ will denote a finite subset of $\mathbb{Z}^\nu$ or a finite set of vertices and $|\Lambda|$ its cardinality. The sequence of finite subsets of $\mathbb{Z}^\nu$ (finite sets of vertices) contained in the sets in a Van Hove sequence/cubelike sequence will itself be called Van Hove sequence/cubelike sequence.

5. THE LYAPUNOV EXPONENT

Consider an ergodic random Schrödinger operator on $l^2(\mathbb{Z})$. Every solution ψ of the eigenvalue equation $H_\omega \psi = E\psi$ associated to (1.1) satisfies

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = A_n(\omega) \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}, \qquad A_n(\omega) = \begin{pmatrix} E - V_n(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

The matrix A_n is called the transfer matrix; recall that $V_n(\omega) = \omega_n$. The subadditive ergodic theorem implies that for all E and for μ-almost all ω

$$\gamma(E) := \lim_{N \to \infty} \frac{1}{N} \log \|A_N(\omega) A_{N-1}(\omega) \cdots A_1(\omega)\| \tag{5.1}$$

exists and is independent of ω (see, e.g., Section 9.3 in ref. 6); $\|\cdot\|$ denotes the operator norm. The quantity γ is called the Lyapunov exponent. Theorem 4.1 can be used to prove a stronger result.

Proposition 5.1. Let $\Omega \subset A^{\mathbb{Z}}$ be defined by a primitive substitution on a finite set A. Then for every $E \in \mathbb{C}$ there exists a $\gamma(E) \in \mathbb{R}$ such that in (5.1) equality holds for all $\omega \in \Omega$.

Proof. Let $E \in \mathbb{C}$ be arbitrary. Since V_n takes finitely many values, there is a $C > 0$ such that $\|A_n(\omega)\| \leqslant C$ for all $\omega \in \Omega$ and all $n \in \mathbb{Z}$. For $\omega \in \Omega$ define a function F_Λ^ω on the finite subsets Λ of $\mathbb{Z}$ by

$$F_\Lambda^\omega := \log \left\| \prod_{i \in \Lambda} A_i(\omega) \right\| - |\Lambda| \, C$$

where the product is taken in descending order of i, as in (5.1). The function F_Λ^ω is subadditive, (4.2), and satisfies

$$-|\Lambda| \, C \leqslant F_\Lambda^\omega \leqslant 0 \qquad \text{for all } \Lambda$$

since $\|A_n(\omega)\| \geqslant 1$ for all n and all ω. If I and I' are equivalent elements of the jth composition $\mathscr{T}^j$ of ω, then $F_I^\omega = F_{I'}^\omega$. Hence Theorem 4.1 gives the existence of an $f^\omega \in [-C, 0]$ such that

$$f^\omega = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} F_{\Lambda_n}^\omega$$

for all cubelike sequences $\{\Lambda_n\}$. Since $\Lambda_n = \{1, 2, ..., n\}$ is a cubelike sequence, this proves that the limit in (5.1) exists for every ω.

It remains to be shown that f^ω does not depend on ω. Let $\Lambda_n = \{1, 2, ..., n\}$. Clearly Λ_n defines a word w_n of length n in ω. Let η be another element of Ω. Since every word in ω also occurs in η, there is an interval Λ_n' of $\mathbb{Z}$ such that the word w_n occurs in η at Λ_n'. Hence $F_{\Lambda_n}^\omega = F_{\Lambda_n'}^\eta$ for all n. This shows that f^ω is independent of ω. ∎

This result is surprising because it is known *not* to be true for uniformly almost-periodic discrete Schrödinger operators. Avron and Simon[21] (Remark 3, p. 389) have shown that for the almost-Mathieu equation the Lyapunov exponent either does not exist for some elements in the hull or is not constant on the hull if the frequency α is a Liouville number. The reason for this difference in behavior is the fact that a sequence generated by a substitution takes finitely many values, whereas a uniformly almost-periodic sequence takes values in an interval. If for a given n two sequences defined by a primitive substitution are sufficiently close in the topology of

Ω, then they will coincide on $\{1, 2,..., n\}$; no such property holds for uniformly almost-periodic sequences.

In addition, it is interesting because it simplifies a step in the proof of Bellisard *et al.*[22] that a Schrödinger operator with "period-doubling potential," which is generated by a primitive substitution, has zero Lebesgue measure for all ω. To prove this, they consider the Lyapunov exponent $\gamma_\omega(E)$ and the constant $\gamma_\mu(E)$ to which it is μ-a.s. equal by the subadditive ergodic theorem and show that the symmetric difference of $\{E \mid \gamma_\mu(E)=0\}$ and $\{E \mid \gamma_\omega(E)=0\}$ has zero Lebesgue measure (Lemma 8). Our result shows that the two sets are equal. Lemma 8 of ref. 22 has also been used in ref. 23.

Proposition 5.1 can be generalized as follows.

Proposition 5.2. Let (Ω, T) be a substitution dynamical system and let B be a function on Ω taking values in the bounded operators on some normed space. Suppose there is a positive integer k such that $B(\omega)$ only depends on $\{\omega_{-k},..., \omega_k\}$. Then

$$\gamma := \lim_{N \to \infty} \frac{1}{N} \log \|B(T^N\omega)\, B(T^{N-1}\omega) \cdots B(T\omega)\| \tag{5.2}$$

exists uniformly in ω.

Sketch of Proof. The proof is analogous to that of Proposition 5.1. One only has to replace the compositions $\mathscr{T}^j$ by "marked compositions" $\mathscr{M}^j$. The elements of $\mathscr{M}^j$ are the same as those in $\mathscr{T}^j$, but elements I_i and I_k of $\mathscr{M}^j$ are equivalent if and only if one has equivalence in $\mathscr{T}^j$ of the pairs I_{i-1} and I_{k-1}, I_i and I_k, and I_{i+1} and I_{k+1}. Theorem 4.1 remains valid if one replaces $\mathscr{T}^j$ by $\mathscr{M}^j$ (see ref. 18 for details). The uniformity follows from, e.g., (4.5). ∎

Walters[24] has investigated the uniform convergence of (5.2) if T is a uniquely ergodic homeomorphism of a compact metrizable space and $B(\omega)$ is an invertible real matrix that depends continuously on ω. He has constructed an example (Theorem 2.2) showing that the convergence is not uniform in general. In addition, he has shown in Theorem 2.1 that the convergence is uniform if all the matrix elements $[B(\omega)]_{ij}$ are strictly positive for all ω. Our Proposition 5.1 shows that this condition is not necessary.

6. SCHRÖDINGER OPERATORS ON PENROSE TILINGS

In this section we prove the existence of the integrated density of states for Schrödinger operators on Penrose tilings. The proof will be given for a large class of bounded self-adjoint operators that contains all the Schrödinger operators that have been studied in the literature. The proof works on all self-similar tilings.

Let L be the set of vertices of a Penrose tiling $\mathcal{T}$. A finite set of vertices will be called a vertexpattern. For positive r, let the r-environment $E_r(\Lambda)$ of a vertex pattern Λ be defined as

$$E_r(\Lambda) := \{x \in L \mid \operatorname{dist}(x, y) \leqslant r \text{ for some } y \in L\}$$

where $\operatorname{dist}(x, y)$ denotes the Euclidean distance between the points x and y; note that $\Lambda \subset E_r(\Lambda)$. Two vertexpatterns Λ and Λ' will be called r-equivalent if $E_r(\Lambda)$ and $E_r(\Lambda')$ are translates of each other.

Consider a bounded self-adjoint operator $H = \{H(x, y)\}_{x, y \in L}$ on $l^2(L)$. Let H_Λ be its restriction to $l^2(\Lambda)$:

$$(H_\Lambda \psi)(x) := \sum_{y \in \Lambda} H(x, y)\, \psi(y)$$

If Λ is a vertexpattern, then H_Λ is a $|\Lambda| \times |\Lambda|$ Hermitian matrix. Two restrictions H_Λ and $H_{\Lambda'}$ can be seen a operators on the same space if Λ and Λ' are translates of each other. The operator H will be called vertexpattern-invariant if there are $r, r' > 0$ such that (i) $H(x, y) = 0$ if $\operatorname{dist}(x, y) > r$ and (ii) $H(x, y) = H(x', y')$ for every $\{x', y'\} \subset L$ such that $E_{r'}(\{x', y'\}) = E_{r'}(\{x, y\}) + a$ for some translation a. If H is a vertexpattern-invariant operator on $l^2(L)$, then there is a vertexpattern-invariant operator H' on the l^2-space of the set of vertices of every other Penrose tiling $\mathcal{T}'$, such that for all vertexpatterns $\Lambda \subset \mathcal{T}$ and $\Lambda' \subset \mathcal{T}'$ one has $H_\Lambda = H_{\Lambda'}$ if Λ and Λ' are translates of each other. In this way, a vertexpattern-invariant operator can be regarded as defined on all Penrose tilings.

Equation (1.1) defines an example of a vertexpattern-invariant operator if two vertices are considered as nearest neighbors if they are connected by an edge of a tile and if the potential $V(x)$ is determined by the type of vertexneighborhood of x. This is the kind of Schrödinger operator that has been considered in refs. 25 and 26 and, with $V(x) \equiv 0$, in refs. 9 and 27–29. The Schrödinger operator in ref. 30 is also vertexpattern-invariant; $H(x, y)$ takes two different values for $x \neq y$ and the potential $H(x, x)$ is zero. Vertexpattern-invariant Schrödinger operators on another aperiodic self-similar tiling, the octagonal tiling, have been studied in refs. 31 and 32.

Boundary conditions can be introduced as follows. Let the inner boundary $\partial^- \Lambda$ of a vertexpattern Λ be the vertexpattern

$$\partial^- \Lambda := \{x \in \Lambda \mid H(x, y) \neq 0 \text{ for some } y \in L \backslash \Lambda\}$$

A boundary condition on Λ is a real function on Λ that vanishes outside $\partial^- \Lambda$. The restriction of H to $l^2(\Lambda)$ with boundary condition χ is the operator H_Λ^χ defined on $l^2(\Lambda)$ by

$$(H_\Lambda^\chi \psi)(x) := \sum_{y \in \Lambda} H(x, y)\, \psi(y) + \chi(x)\, \psi(x)$$

Let $N_\Lambda^\chi(\lambda)$ be the number of eigenvalues of H_Λ^χ, counting multiplicity, less than or equal to λ. If $\chi \geqslant \chi'$, then $H_\Lambda^\chi \geqslant H_\Lambda^{\chi'}$ and monotonicity (see, e.g., Corollary 4.3.3 in ref. 33) implies that $N_\Lambda^\chi(\lambda) \leqslant N_\Lambda^{\chi'}(\lambda)$ for all λ. Since $0 \leqslant N_\Lambda^\chi(\lambda) \leqslant |\Lambda|$ for every boundary condition χ and all Λ, it is possible to consider boundary conditions "∞" and "$-\infty$." We shall write N_Λ^∞ and $N_\Lambda^{-\infty}$ for the limit of N_Λ^χ when all components of χ tend to ∞ and $-\infty$, respectively. The limit of $|\Lambda|^{-1} N_\Lambda$ as $|\Lambda| \to \infty$ is known as the integrated density of states.

Proposition 6.1. The integrated density of states exists for every self-adjoint vertexpattern-invariant operator and is independent of boundary conditions and the Penrose tiling under consideration: there exists a function $N(\lambda)$ such that on every Penrose tiling

$$N(\lambda) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} N_{\Lambda_n}^{\chi_n}(\lambda) \qquad \text{for all } \lambda$$

for every cubelike sequence $\{\Lambda_n\}$ and every sequence of boundary conditions χ_n, which may be ∞ or $-\infty$.

Proof. We shall use Theorem 4.1 to show that the limit exists along cubelike sequences on a given Penrose tiling. The fact that every patch occurring in one Penrose tiling occurs in every Penrose tiling implies that the limit is the same on every Penrose tiling (cf. the end of the proof of Proposition 5.1).

By using the "χ-bracketing" of ref. 34, it is not difficult to show that

$$N_{\Lambda_1}^\infty(\lambda) + N_{\Lambda_2}^\infty(\lambda) \leqslant N_{\Lambda_1 \cup \Lambda_2}^\infty(\lambda) \qquad \text{for all } \lambda, \quad \text{if } \Lambda_1 \cap \Lambda_2 = \varnothing$$

Thus $-N_\Lambda^\infty$ is a subadditive set function. Since H is vertexpattern-invariant, there exists an $r' > 0$ such that $N_\Lambda^\infty = N_{\Lambda'}^\infty$ if Λ and Λ' are r'-equivalent. In a Penrose tiling, two fat (skinny) rhombs in $\mathscr{T}^j$ have (for $j \geqslant 5$) the same environment of tiles from $\mathscr{T}$ up to a distance that grows propertional to τ^j. Therefore $N_V^\infty = N_{V'}^\infty$ on equivalent tiles and vertexneighborhoods of $\mathscr{T}^j$ if j is sufficiently large (on an arbitrary self-similar tilings one has this equality for equivalent sets and vertex-neighborhoods of a "marked" partition[18]). Thus Theorem 4.1 gives the existence of $\lim_{|\Lambda| \to \infty} |\Lambda|^{-1} N_\Lambda^\infty(\lambda)$, for every λ, along cubelike sequences.

The fact that the limit is independent of the boundary conditions follows from the observation that for every vertexpattern Λ and every pair of boundary conditions χ and χ' one has

$$|N_\Lambda^\chi(\lambda) - N_\Lambda^{\chi'}(\lambda)| \leqslant |\partial^- \Lambda| \qquad \text{for all } \lambda \tag{6.1}$$

To prove this estimate, note that H_Λ^χ and $H_\Lambda^{\chi'}$ differ by a Hermitian matrix of rank at most $r := |\partial^- \Lambda|$. If A and B are Hermitian $n \times n$ matrices and B has rank at most r, then (see, e.g., Theorem 4.3.6 in ref. 33)

$$\lambda_k(A+B) \leqslant \lambda_{k+r}(A) \leqslant \lambda_{k+2r}(A+B) \qquad \text{for} \quad k = 1, 2, ..., n-2r$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ are the ordered eigenvalues of the matrix (the same inequalities hold with A and $A+B$ interchanged). These inequalities readily imply (6.1). ∎

We conclude this section with some miscellaneous remarks. First, (6.1) also shows that periodic boundary conditions do not affect the integrated density of states in the thermodynamic limit. Here, "periodic boundary conditions" can mean several things. For instance, one might take the vertices inside a cube, repeat this pattern periodically, and use an *ad hoc* definition of $H(x, y)$ for x and y in neighboring cubes. Or one could consider a "periodic approximant" to the tiling, if the tiling can also be described by the "projection method," as is the case for Penrose tilings and the octagonal tilings. Second, it is by no means essential that the operators act on the l^2-space of the set of vertices. The argument also applies if L is the set of points one gets by putting a finite number of "atoms" on every tile, provided all fat (skinny) rhombs are decorated in the same way. In particular, Proposition 6.1 holds for the Schrödinger operator on the dual lattice of the Penrose tiling discussed in ref. 35. Third, the assumption that $H(x, y)$ is of finite range is not essential either; it suffices that it decays fast enough as $\operatorname{dist}(x, y) \to \infty$ that the set function N_Λ satisfies (4.3). Finally, recall that sequences generated by primitive substitutions have a self-similarity property analogous to that of the Penrose tilings. Therefore Proposition 6.1 has an analog for one-dimensional Schrödinger operators with potential generated by a primitive substitution. In the next section we prove the existence of the integrated density of states and its independence of the realization of the potential by the strict ergodicity of the substitution dynamical system.

7. STRICTLY ERGODIC SCHRÖDINGER OPERATORS

7.1. Introduction and Notation

This section discusses Schrödinger operators H_ω on $l^2(\mathbb{Z}^\nu)$ of the form (1.1)—and satisfying (1.2)—where $V_\omega(n) = \omega_n$ and ω is an element of a space Ω that is compact, and strictly ergodic under translations. Strict ergodicity is defined below. The next subsection states and proves the results. The last subsection gives examples of strictly ergodic systems and discusses the extent to which the results are new.

Let Ω be a compact metrizable space and let $\mathscr{F}$ be its Borel σ-algebra. A $\mathbb{Z}^\nu$-action on Ω is a family of homeomorphisms $\{T_n\}_{n \in \mathbb{Z}^\nu}$ on Ω such that T_0 is the identity on Ω and $T_n(T_m\omega) = T_{n+m}\omega$ for all $n, m \in \mathbb{Z}^\nu$ and all $\omega \in \Omega$. A set $A \in \mathscr{F}$ is called invariant if $T_n A = A$ for all $n \in \mathbb{Z}^\nu$. A probability measure μ on $(\Omega, \mathscr{F})$ is called invariant if $\mu(A) = \mu(T_n A)$ for all $A \in \mathscr{F}$ and all $n \in \mathbb{Z}^\nu$. A measure μ on $(\Omega, \mathscr{F})$ is called ergodic if it is an invariant probability measure and $\mu(A)$ is either 0 or 1 for every invariant set $A \in \mathscr{F}$. The orbit $\mathrm{Orb}(\omega)$ of $\omega \in \Omega$ is the set $\{T_n\omega\}_{n \in \mathbb{Z}^\nu} \subset \Omega$. If there is only one ergodic measure on Ω, then Ω (or the probability measure, or the $\mathbb{Z}^\nu$-action, or the dynamical system) is called uniquely ergodic, it is called minimal if $\mathrm{Orb}(\omega)$ is dense in Ω for every $\omega \in \Omega$, and it is called strictly ergodic if it is both minimal and uniquely ergodic. Below, let μ be the uniquely ergodic probability measure on Ω.

We shall need the following results. For proofs in dimension one, see, e.g., Section 6.5 in ref. 36. It is straightforward to generalize the proofs to higher dimensions.

Proposition 7.1. The following two statements are equivalent:

(i) Ω is uniquely ergodic.

(ii) For every continuous function f on Ω

$$\lim_{n \to \infty} n^{-\nu} \sum_{x \in C_n} f(T_x\omega) = \int f \, d\mu \tag{7.1}$$

uniformly in $\omega \in \Omega$, where $\{C_n\}$ is a sequence of cubes of side n.

Proposition 7.2. If Ω is uniquely ergodic, then Ω is minimal if and only if $\mu(A) > 0$ for every open subset A of Ω.

In statement (ii) of Proposition 7.1 the sequence $\{C_n\}$ can be replaced by a Van Hove sequence. In what follows Ω will be a closed invariant subset of $E^{\mathbb{Z}^\nu}$, where E is a compact subset of $\mathbb{R}$, and $(T_n\omega)_k := \omega_{k+n}$. The topology of Ω is specified in the examples in Section 7.3.

7.2. Results

Let χ_L denote the characteristic function of the cube C_L of side $2L+1$ in $\mathbb{Z}^\nu$ centered around the origin. Let $E_A(\omega)$ denote the resolution of the identity of H_ω. Denote by δ_n the element of $l^2(\mathbb{Z}^\nu)$ such that $(\delta_n)_k = \delta_{nk}$, where δ_{nk} is the Kronecker delta. Define a measure dk_L^ω on $\mathbb{R}$ by

$$\int_A dk_L^\omega := (2L+1)^{-\nu} \operatorname{tr}(E_A(\omega)\,\chi_L)$$

We shall show that for all ω the measures dk_L^ω converge vaguely as $L \to \infty$ to a measure dk given by

$$\int f(\lambda)\, dk(\lambda) = \int (\delta_0, f(H_\omega)\,\delta_0)\, d\mu(\omega) \tag{7.2}$$

It is the unique ergodicity that makes the convergence hold for all $\omega \in \Omega$; for ergodic Schrödinger operators the convergence holds almost everywhere (see, e.g., Theorem 9.7 in ref. 6). The integrated density of states is the distribution function of dk.

Proposition 7.3. If Ω is uniquely ergodic, then $dk_L^\omega \to dk$ vaguely as $L \to \infty$ for all $\omega \in \Omega$.

Proof. We have to show that for every bounded continuous function of compact support

$$\lim_{L\to\infty} \int f\, dk_L^\omega = \int f\, dk$$

for all $\omega \in \Omega$, where dk is given by (7.2).

Let f be a bounded continuous function of compact support and define a function $\bar{f}$ on Ω by $\bar{f}(\omega) = (\delta_0, f(H_\omega)\,\delta_0)$. Clearly, $\bar{f}(T_n\omega) = (\delta_n, f(H_\omega)\,\delta_n)$ for all $n \in \mathbb{Z}^\nu$. If $\omega_n \to \omega$ in Ω, then $H_{\omega_n} \to H_\omega$ strongly. Therefore, $f(H_{\omega_n}) \to f(H_\omega)$ strongly (see, e.g., Theorem X.7.1 in ref. 37). Hence $\bar{f}$ is a continuous function on Ω.

Now

$$\begin{aligned}\int f(\lambda)\, dk_L^\omega(\lambda) &= (2L+1)^{-\nu} \operatorname{tr}(f(H_\omega)\,\chi_L)\\ &= (2L+1)^{-\nu} \sum_{n\in C_L} (\delta_n, f(H_\omega)\,\delta_n)\\ &= (2L+1)^{-\nu} \sum_{n\in C_L} \bar{f}(T_n\omega)\end{aligned}$$

As $L \to \infty$, the last expression converges by Proposition 7.1 to $\int \bar{f}\, d\mu$ uniformly in ω. ∎

Proposition 7.4. If Ω is strictly ergodic, then for all $\omega \in \Omega$ the spectrum Σ_ω of H_ω coincides with the topological support of dk. In particular, Σ_ω is independent of ω.

Proof. If $\lambda_0 \notin \Sigma_\omega$, there is a nonnegative continuous function of compact support such that $f(\lambda_0) = 1$ and $f(\lambda) = 0$ for $\lambda \in \Sigma_\omega$. Therefore $f(H_\omega)$ is the zero operator and $\bar{f}(\omega) \equiv 0$ (the function $\bar{f}$ is defined in the proof of Proposition 7.3). Since $\int \bar{f}\, d\mu = \int f\, dk$ by (7.2), $\lambda_0 \notin \operatorname{supp}(dk)$. This proves that $\operatorname{supp}(dk) \subset \Sigma_\omega$ for all ω.

To prove that $\Sigma_\omega \subset \operatorname{supp}(dk)$ for all ω, we show that $\lambda_0 \notin \operatorname{supp}(dk)$ implies that $\lambda_0 \notin \Sigma_\omega$ for all ω. If $\lambda_0 \notin \operatorname{supp}(dk)$, there is a continuous function f of compact support such that $f(\lambda_0) = 1$ and $\int f(\lambda)\, dk(\lambda) = 0$. Then, by (7.2), $\int \bar{f}\, d\mu = 0$. Since f is nonnegative, $f(H_\omega) \geqslant 0$ and hence $\bar{f}(\omega) \geqslant 0$ for all ω. The minimality of Ω and the continuity of $\bar{f}$ imply by Proposition 7.2 that $\bar{f}(\omega) \equiv 0$. In particular, $\bar{f}(T_n \omega) = (\delta_n, f(H_\omega)\, \delta_n) = 0$ for all $n \in \mathbb{Z}^\nu$ and all $\omega \in \Omega$. As $f(H_\omega) \geqslant 0$, this implies $f(H_\omega) = 0$ for all ω. Since f is continuous and $f(\lambda_0) = 1$, it follows that $\lambda_0 \notin \Sigma_\omega$, for all $\omega \in \Omega$. ∎

The first paragraph of this proof is identical to the proof of Proposition 9.8 in ref. 6; the second differs from it in that it uses Proposition 7.2. Note that Propositions 7.3 and 7.4 hold for arbitrary bounded self-adjoint operators H_ω on $l^2(\mathbb{Z}^\nu)$ that satisfy (1.2) and are strongly continuous in ω.

7.3. Examples

Below we describe three classes of strictly ergodic systems: systems defined by substitutions, by "circle maps," and by uniformly almost-periodic functions. In the first two classes, E is a finite set with the discrete metric and Ω has the product topology. In the last class E is an interval and Ω is equipped with the supremum norm. The examples are followed by a discussion of the extent to which Propositions 7.3 and 7.2 are new for each class.

As stated in Section 2, every primitive substitution defines a strictly ergodic dynamical system. One-dimensional Schrödinger operators with potential defined by a primitive substitution have been studied in, e.g., refs. 38, 22, 39, and 23. But, as explained in Section 3, one can also consider higher-dimensional primitive substitutions, and these give strictly ergodic dynamical systems, too. The direct products of Fibonacci sequences considered in refs. 30 and 40–42 fall into this category.

The potential of a one-dimensional Schrödinger operator is said to be defined by a circle map if $\omega_n = 1_{[0,\beta)}(\theta + n\alpha)$, where $\theta + n\alpha$ is considered modulo 1, or as an element of the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $0 < \beta < 1$. Schrödinger operators of this kind are discussed in, e.g., refs. 43–48. Actually, most papers take $\alpha = \beta$. For $\alpha = \beta = \frac{1}{2}(\sqrt{5} - 1)$ the sequence ω is a Fibonacci sequence. It seems not to have been realized that these Schrödinger operators are strictly ergodic if α is irrational. They even remain strictly ergodic if $[0, \beta)$ is replaced by a countable union of disjoint half-open intervals (see Proposition A.1). One can make strictly ergodic systems in which sequences take N values by partitioning $[0, 1)$ into N sets that are countable unions of half-open intervals.

Note that for the minimality it is essential that the intervals are half open. If $\omega_n = 1_{[0,\alpha]}(\theta + n\alpha)$, the sequence contains two consecutive 1's if $\theta + k\alpha = 0$ for some integer k ($\omega_k = \omega_{k+1} = 1$), but for all other values of θ the sequence contains only isolated 1's.

The construction can be extended to arbitrary dimension by replacing the circle by a ν-torus. The ν-torus with $\mathbb{Z}^\nu$-action defined by translations $\alpha_1, \ldots, \alpha_\nu$ is strictly ergodic if and only if the α_i are rationally independent, i.e., if $k_1\alpha_1 + \cdots + k_\nu\alpha_\nu = 0$ has $k_1 = \cdots = k_\nu = 0$ as the only solution in the integers (this can, e.g., be shown by a straightforward modification of the proofs of Theorems 3.1.1 and 3.1.2 in ref. 49). If a half-open interval is now taken to be a Cartesian product of intervals $[a_i, b_i)$, the analog of Proposition A.1 is easily seen to hold.

The last class of examples is provided by uniformly almost-periodic functions. A continuous function $f\colon \mathbb{R}^\nu \to \mathbb{R}^\nu$ is called uniformly almost-periodic if the set of its translates $\{f(\cdot + s)\}_{s \in \mathbb{R}^\nu}$ has compact closure in the $\|\cdot\|_\infty$-topology. A uniformly almost-continuous function has a compact range. Hence, if f is a uniformly almost-periodic function and $\omega_n = f(n)$, then the orbit closure Ω of ω in the $\|\cdot\|_\infty$-topology is $\|\cdot\|_\infty$-compact. The minimality follows from the fact that for every $\eta, \eta' \in \Omega$ there are translates of ω having distance less then $\varepsilon/2$ to η and η', respectively; the triangle inequality then gives that there is a translate of η having a distance of less than ε to η'. It is well known that Ω is uniquely ergodic (see, e.g., Section 10.1 in ref. 6). Schrödinger operators with uniformly almost-periodic potential have been studied extensively (for reviews see, e.g., refs. 6–8).

Sequences defined by primitive substitutions or circle maps are also often called almost-periodic or quasiperiodic. It should be noted that they are almost-periodic in the sense that every word occurs with bounded intervals (ref. 10, p. 71). These sequences are not uniformly almost-periodic.

The facts that the integrated density of states and the spectrum are independent of the realization of the potential are known for many of these examples. Both facts are well known for uniformly almost-periodic

Schrödinger operators in arbitrary dimension, but the proof usually depends on the inequality $\|H_\omega - H_\eta\| \leqslant \|\omega - \eta\|_\infty$. Bellisard *et al.*[22] observe on p. 398 that in one dimension unique ergodicity implies that the integrated density of states is independent of ω; their argument is different from ours. As said before, it seems not to have been noticed that circle maps give uniquely ergodic systems for irrational α. It has been shown, however, that the spectrum of circle-map Schrödinger operators with $\alpha = \beta$ is independent of θ (Lemma 3 in ref. 47). And implicitly the proof of that lemma shows that minimality implies that the spectrum is independent of the realization of the potential.

What is new in Propositions 7.3 and 7.4 is that they hold in arbitrary dimension and that their abstract formulation permits a unified treatment of the three classes of examples. They show explicitly that the integrated density of states is constant for circle maps and that the spectrum is constant for primitive substitutions. In addition, some of the examples are new. Apart from the direct products in refs. 30 and 40–42, higher-dimensional substitutions have, as far as we know, not been considered in the literature, nor have higher-dimensional circle maps or the circle maps in the general form of Proposition A.1.

APPENDIX

Proposition A.1. Consider the torus $\mathbb{T} = [0, 1)$, an element $\theta \in \mathbb{T}$, an irrational number α, and the translation $x \to x + \alpha$ on $\mathbb{T}$. Let A_1 be a countable union of disjoint half-open intervals $[a, b)$ in $\mathbb{T}$, let ϕ be its characteristic function, and $A_0 := \mathbb{T} \backslash A_1$. Define $\omega^\theta \in \{0, 1\}^{\mathbb{Z}}$ by $\omega_n^\theta := \phi(\theta + n\alpha)$. Then:

(a) The closure Ω in $\{0, 1\}^{\mathbb{Z}}$ of the orbit of ω^θ under the shift T is independent of θ.

(b) (Ω, T) is strictly ergodic.

Proof. Let $B = B_0 \cdots B_{l-1}$ be a word of length l that occurs in ω^θ at k, i.e., $\omega_{k+j}^\theta = B_j$ for $0 \leqslant j < l$. Equivalently, $\theta + k\alpha \in \bigcap_{j=0}^{l-1} \{\phi^{-1}(B_j) - j\alpha\}$ $=: C$. If the intersection of $[a_1, b_1)$ and $[a_2, b_2)$ is not empty, then it is of the form $[a_3, b_3)$. Therefore, and since $\phi^{-1}(B_j) = A_j$, the set C is again a countable union of disjoint half-open intervals. It contains an open set because it is not empty. Since translation over an irrational number is ergodic (even uniquely ergodic; see, e.g., Theorem 3.1.2 in ref. 49), the word B occurs infinitely often in ω^θ, and also infinitely often in $\omega^{\theta'}$ for every $\theta' \in \mathbb{T}$. This implies (a).

To show that (Ω, T) is uniquely ergodic, note that the characteristic functions of words are $\|\cdot\|_\infty$-dense in the set of all continuous functions on Ω. Hence by Proposition 7.1 it suffices to show that every word occurs in ω^θ with a uniformly defined frequency. So let the word B and the set C be as above. Since C is a countable union of intervals, there exist for every $\varepsilon > 0$ continuous functions f_1 and f_2 on $\mathbb{T}$ such that $f_1 \leqslant 1_C \leqslant f_2$ and $\int f_2 - f_1 \, d\lambda \leqslant \varepsilon$, where $d\lambda$ denotes the normalized Lebesgue measure on $\mathbb{T}$. Since translation over α is uniquely ergodic, Proposition 7.1 gives that for all integers k

$$\begin{aligned}\int f_1 \, d\lambda &\leqslant \liminf_{n \to \infty} \frac{1}{n} \sum_{j=k}^{k+n-1} 1_C(\theta + j\alpha) \\ &\leqslant \limsup_{n \to \infty} \frac{1}{n} \sum_{j=k}^{k+n-1} 1_C(\theta + j\alpha) \leqslant \int f_2 \, d\lambda\end{aligned}$$

Since ε is arbitrary, this means that B occurs with a uniform frequency in ω^θ, and this frequency is equal to the Lebesgue measure of C. Since C contains an open set, this frequency is strictly positive. Proposition 7.2 now gives that (Ω, T) is minimal and hence strictly ergodic. This proves (b). ∎

ACKNOWLEDGMENTS

It is a pleasure to thank the IHES, where this work was completed, for its hospitality, and J.-M. Ghez, B. Iochum, O. Knill, and A.-S. Sznitman for discussions.

REFERENCES

1. U. Mizutani, Y. Sakabe, T. Shibuya, K. Kishi, K. Kimura, and S. Takeuchi, Electron transport properties of thermodynamically stable Al–Cu–Ru icosahedral quasicrystals, *J. Phys.: Cond. Matter* **2**:6169–6178 (1990).
2. T. Klein, A. Gozlan, C. Berger, F. Cyrot-Lackmann, Y. Calvayrac, and A. Quivy, Anomalous transport properties in pure AlCuFe icosahedral phases of high structural quality, *Europhys. Lett.* **13**:129–134 (1990).
3. B. D. Biggs, S. J. Poon, and N. R. Munirathnam, Stable Al–Cu–Ru icosahedral crystals: A new class of electronic alloys, *Phys. Rev. Lett.* **65**:2700–2703 (1990).
4. D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, Metallic phase with long-range orientational order and no translational symmetric, *Phys. Rev. Lett.* **53**:1951–1953 (1984).
5. J. Bellisard, Almost periodicity in solid state physics and C^*-algebras, in *The Harald Bohr Centenary*, C. Berg and F. Flugede, eds., *Dansk. Vid. Selsk. Mat.-Fys. Medd.* **42**(3):35–75 (1989).
6. H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators—with Application to Quantum Mechanics and Global Geometry* (Springer-Verlag, Berlin, 1987).
7. R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators* (Birkhäuser, Basel, 1990).

8. L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators* (Springer-Verlag, Berlin, 1992).
9. M. Kohmoto and B. Sutherland, Electronic states on a Penrose lattice, *Phys. Rev. Lett.* **56**:2740–2743 (1986).
10. M. Queffélec, *Substitution Dynamical Systems—Spectral Analysis*, Lect. Notes in Math. 1294 (Springer, 1987).
11. F. M. Dekking, The spectrum of dynamical systems arising from substitutions of constant length, *Z. Wahr. Verw. Geb.* **41**:221–239 (1978).
12. A. Hof, Quasicrystals, aperiodicity and lattice systems, Thesis, University of Groningen, The Netherlands (1992).
13. W. F. Lunnon and P. A. B. Pleasants, Quasicrystallographic tilings, *J. Math. Pures Appl.* **66**:217–263 (1987).
14. B. Grünbaum and G. C. Shephard, *Tilings and Patterns* (Freeman, 1987).
15. N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings of the plane. I, *Ned. Akad. Wetensch. Proc. A* **84**:39–52 (1981) (=*Indag. Math.* **84**:39–52).
16. N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings of the plane. II, *Ned. Akad. Wetensch. Proc. A* **84**:53–66 (1981) (=*Indag. Math.* **84**:53–66).
17. N. G. de Bruijn, Updown generation of Penrose tilings, *Indag. Math. N. S.* **1**:201–220 (1990).
18. C. P. M. Geerse and A. Hof, Lattice gas models on self-similar aperiodic tilings, *Rev. Math. Phys.* **3**:163–221 (1991).
19. D. W. Robinson and D. Ruelle, Mean entropy of states in classical statistical mechanics, *Commun. Math. Phys.* **5**:288–300 (1967).
20. M. E. Fisher, The free energy of a macroscopic system, *Arch. Rat. Mech.* **17**:377–410 (1964).
21. J. Avron and B. Simon, Almost periodic Schrödinger operators II. The integrated density of states, *Duke Math. J.* **50**:369–391 (1983).
22. J. Bellisard, A. Bovier, and J.-M. Ghez, Spectral properties of a tight binding Hamiltonian with period doubling potential, *Commun. Math. Phys.* **135**:379–399 (1991).
23. A. Bovier and J.-M. Ghez, Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions, Preprint CNRS Luminy (1992).
24. P. Walters, Unique ergodicity and random matrix products, in *Lyapunov Exponents—Proceedings, Bremen, 1984*, L. Arnold and V. Wihstutz, eds., Lect. Notes in Math. 1186 (Springer-Verlag, Berlin, 1986), pp. 37–55.
25. M. Kohmoto and B. Sutherland, Electronic and vibrational modes on a Penrose lattice: Localized states and band structure, *Phys. Rev. B* **34**:3849–3853 (1986).
26. B. Sutherland, Self-similar ground state wave function for electrons on a two-dimensional Penrose lattice, *Phys. Rev. B* **34**:3904–3909 (1986).
27. T. Odagaki and D. Nguyen, Electronic and vibrational spectra of two-dimensional quasicrystals, *Phys. Rev. B* **33**:2184–2190 (1986); Erratum, *Phys. Rev. B* **34**:5929–5930 (1986).
28. M. Arai, T. Tokihiro, and T. Fujiwara, Strictly localized states on a two-dimensional Penrose lattice, *Phys. Rev. B* **38**:1621–1626 (1988).
29. Y. Liu and P. Ma, Electronic properties of two-dimensional quasicrystals with near-neighbour interactions, *Phys. Rev. B* **43**:1378–1384 (1991).
30. J. A. Ashraff, J.-M. Luck, and R. B. Stinchcombe, Dynamical properties of two-dimensional quasicrystals, *Phys. Rev. B* **41**:4314–4329 (1990).
31. V. G. Benza and C. Sire, Band spectrum of the octagonal quasicrystal: Finite measure, gaps, and chaos, *Phys. Rev. B* **44**:10343–10345 (1991).
32. B. Passaro, C. Sire, and V. G. Benza, Anomalous diffusion and conductivity in octagonal tiling models, *Phys. Rev. B* **46**:13751–13755 (1992).

33. R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1985).
34. G. A. Mezincescu, Lifschitz singularities for periodic operators plus random potentials, *J. Stat. Phys.* **49**:1181–1190 (1987).
35. H. Tsunetsugu, F. Fujiwara, K. Ueda, and T. Tokihiro, Electronic properties of the Penrose lattice. I. Energy spectrum and wave functions, *Phys. Rev. B* **43**:8879–8891 (1991).
36. P. Walters, *An Introduction to Ergodic Theory* (Springer-Verlag, Berlin, 1982).
37. N. Dunford and J. T. Schwartz, *Linear Operators. Part II: Spectral Theory—Self-Adjoint Operators in Hilbert Space* (Interscience, New York, 1963).
38. J. M. Luck, Cantor spectra and scaling of gap widths in deterministic aperiodic systems, *Phys. Rev. B* **39**:5834–5849 (1989).
39. J. Bellisard, A. Bovier, and J.-M. Ghez, Gap labelling theorems for one dimensional discrete Schrödinger operators, *Rev. Math. Phys.* **4**:1–37 (1992).
40. K. Ueda and H. Tsunetsugu, Energy spectrum and conductance of a two-dimensional quasicrystal, *Phys. Rev. Lett.* **58**:1272–1275 (1987).
41. W. A. Schwalm and M. K. Schwalm, Extension theory for lattice Green functions, *Phys. Rev. B* **37**:9524–9542 (1988).
42. X. Fu, Y. Liu, B. Cheng, and D. Zheng, Spectral structure of two-dimensional Fibonacci quasilattices, *Phys. Rev. B* **43**:10808–10814 (1991).
43. J. Bellisard and E. Scoppola, The density of states for almost periodic Schrödinger operators and the frequency module: A counter example, *Commun. Math. Phys.* **85**:301–308 (1982).
44. M. Kohmoto, L. P. Kadanoff, and C. Tang, Localization problem in one dimension: Mapping and escape, *Phys. Rev. Lett.* **50**:1870–1872 (1983).
45. F. Delyon and D. Petritis, Absence of localization in a class of Schrödinger operators with quasiperiodic potential, *Commun. Math. Phys.* **103**:441–444 (1986).
46. M. Casdagli, Symbolic dynamics for the renormalization map of a quasiperiodic Schrödinger equation, *Commun. Math. Phys.* **107**:295–318 (1986).
47. J. Bellisard, B. Iochum, E. Scoppola, and D. Testard, Spectral properties of one dimensional quasi-crystals, *Commun. Math. Phys.* **125**:527–543 (1989).
48. J. Bellisard, B. Iochum, and D. Testard, Continuity properties of the electronic spectrum of 1d quasicrystals, *Commun. Math. Phys.* **141**:353–380 (1991).
49. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer-Verlag, Berlin, 1982).